Toric Varieties

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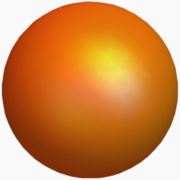
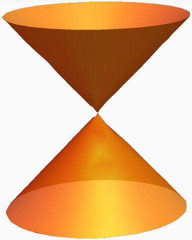
Algebraic varieties

$x^2 + y^2 = 1$



$x^2 + y^2 + z^2 = 1$

$y^2 = x^3 + 2x^2$



$xy = z^2$

What are toric varieties, and why study them?

Toric varieties are a special class of varieties determined by elementary convex geometry.

They serve as testing grounds for conjectures and theorems in algebraic geometry.

For example, the Hodge conjecture—a Millennium Prize Problem—is proven for toric varieties but unresolved in general.

Goal of the talk: introduce toric varieties and explain some of their properties.

The definition of a variety

Varieties are sets of solutions $(a_1, \ldots, a_n) \in \mathbb{C}^n$ to poly. equations

$$f_1(a_1, \ldots, a_n) = 0, \quad \ldots, \quad f_s(a_1, \ldots, a_n) = 0.$$

Choose the zero polynomial:

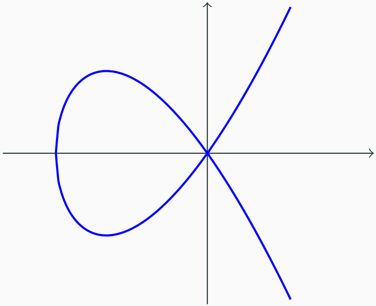
$$\rightsquigarrow \mathbb{C}^n.$$

Choose $y^2 = x^3 + 2x^2$:

$$\rightsquigarrow \text{singular curve}.$$

Choose $xy = 1$:

$$\rightsquigarrow \{(t, t^{-1}) : t \in \mathbb{C}^\times\} \cong \mathbb{C}^\times.$$



$\mathbb{C}^\times$ is called the algebraic torus. The d-dimensional torus is $(\mathbb{C}^\times)^d$.

Convex cones

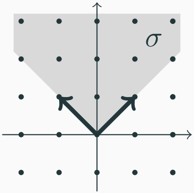
To study toric varieties, we need to understand convex cones.

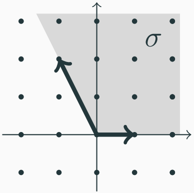
We consider polyhedral cones. These are sets in $\mathbb{R}^n$ of the form

$$\sigma = \operatorname{span}_{\mathbb{R}_{\geq 0}}\{v_1, \ldots, v_r\},$$

for some $v_1, \ldots, v_r \in \mathbb{R}^n$.

We call σ rational if we can take each $v_i \in \mathbb{Z}^n$.

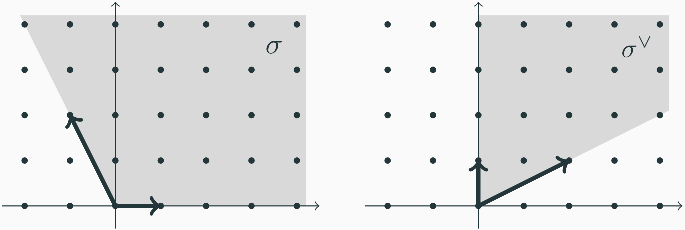




Dual cones

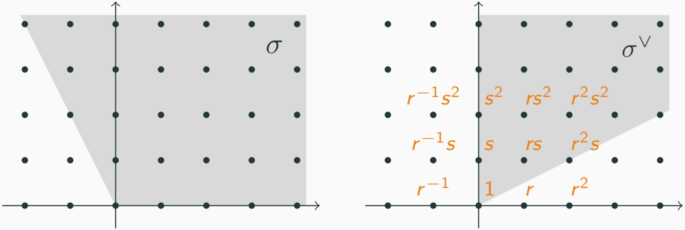
Fix: cone σ. The dual cone is the set of linear functionals which are non-negative on σ:

$$\sigma^\vee := \{u \in (\mathbb{R}^n)^* : u(v) \geq 0 \text{ for all } v \in \sigma\}.$$



An example of a toric variety

Fix: cone σ, dual $\sigma^\vee$. Monomials live on integer points in $(\mathbb{R}^n)^*$:



We create a ring using the monomials in $\sigma^\vee$:

$$\begin{aligned}\mathbb{C}[1, s, r^2s, rs, s^2, rs^2, \dots] &= \mathbb{C}[s, r^2s, rs] \\ &\cong \mathbb{C}[x, y, z]/(xy - z^2).\end{aligned}$$

The toric variety U_σ is the set of solutions to $xy - z^2 = 0$ in $\mathbb{C}^3$:

$$xy = z^2.$$

The definition of a toric variety

Fix: cone σ, dual $\sigma^\vee$. The previous construction generalises.

We associate monomials $x_1^{i_1} \cdots x_n^{i_n}$ to integer points in $(\mathbb{R}^n)^*$.

We create a ring using the monomials lying in $\sigma^\vee$, called R_σ.

R_σ is finitely generated, so it's given by generators and relations:

$$R_\sigma = \mathbb{C}[y_1, \ldots, y_m]/(f_1, \ldots, f_s).$$

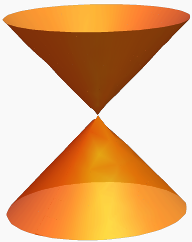
The toric variety U_σ is the subset of $\mathbb{C}^m$ defined by the equations

$$f_1(a_1, \ldots, a_m) = 0, \quad \ldots, \quad f_s(a_1, \ldots, a_m) = 0.$$

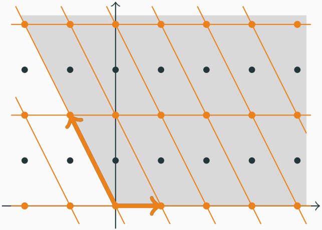
Cones detect singularities

Theorem

The toric variety U_σ is non-singular if and only if σ is generated by a subset of a basis for $\mathbb{Z}^n$.



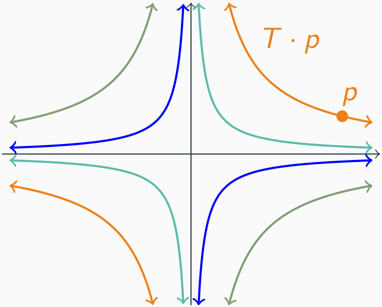
$xy = z^2$



Torus quotients

Suppose $T = (\mathbb{C}^\times)^d$ acts linearly on $\mathbb{C}^n$. **Goal**: understand $\mathbb{C}^n/T$.

Example: $\mathbb{C}^\times \curvearrowright \mathbb{C}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$.



Problem: determine invariant polynomials

$$\mathbb{C}[x, y]^T = \{\text{polys } f : f(p) = f(T \cdot p)\}.$$

Torus quotients as toric varieties

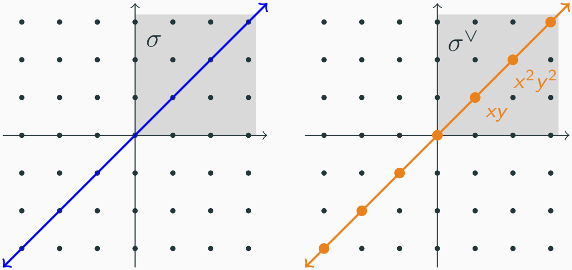
Theorem

$\mathbb{C}^n/T$ is a toric variety whose cone can be explicitly computed.

Idea: Invariant monomials are parametrised by points in $\mathbb{Z}^n$ satisfying linear equations. For $\mathbb{C}^\times \curvearrowright \mathbb{C}^2$:

$$x^k y^\ell = t^{k-\ell} x^k y^\ell \qquad \iff \qquad k = \ell.$$

Choose σ so $\sigma^\vee$ only sees points with non-negative coordinates.



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