

POLYNOMIAL REPRESENTATIONS OF $\mathrm{GL}_n(\mathbb{C})$

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1. INTRODUCTION

Although the beginner representation theorist often first learns about the representation theory of finite groups, the representation theory of *Lie groups* is a deep and beautiful subject. A prototypical example in this theory is that of the polynomial representations of $\mathrm{GL}_n(\mathbb{C})$. In this report, we will detail a construction of the polynomial representations of $\mathrm{GL}_n(\mathbb{C})$ that originated in Deryuts's 1891 treatise *Essai d'une théorie générale des formes algébriques* [2]. Like the representation theory of the symmetric group, the construction of these modules is intimately connected with the combinatorics of Young tableaux. After seeing examples of these modules, we will investigate their structure further via the weight space decomposition. We conclude by computing the characters of these representations.

2. POLYNOMIAL REPRESENTATIONS

Throughout, let GL_n denote $\mathrm{GL}_n(\mathbb{C})$. Recall that a GL_n -module is an action of GL_n on a complex vector space V . After fixing a basis for V , a GL_n -module is equivalent to a group homomorphism

$$\rho: \mathrm{GL}_n \rightarrow \mathrm{GL}_N,$$

where N is the dimension of V .

A fundamental question in the representation theory of a group G is: *what are all representations of G ?* This question has been answered decisively for the symmetric group; all representations are a direct sum of irreducibles, and the *Specht modules* give the complete list of irreducible representations. When we turn from finite groups like the symmetric group to Lie groups, the goal of classifying *all* representations is far more ambitious. Consequently, it makes sense to restrict our attention to particular “nice” representations. For GL_n , one such class of nice representations is the *polynomial representations*.

Definition 2.1. A representation $\rho: \mathrm{GL}_n \rightarrow \mathrm{GL}_N$ is said to be polynomial if the entries of $\rho(g)$ are polynomial functions in the entries of g . Specifically, if $g = (g_{ij})_{1 \leq i, j \leq n}$ and $\rho(g) = (\rho(g)_{kl})_{1 \leq k, l \leq N}$, there must exist polynomials $p_{kl} \in \mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}]$ such that

$$\rho(g)_{kl} = p_{kl}(g_{11}, g_{12}, \dots, g_{nn})$$

for all $g \in \mathrm{GL}_n$ and $1 \leq k, l \leq N$.

Importantly, this notion is independent of the choice of basis for V . Say $\rho: \mathrm{GL}_n \rightarrow \mathrm{GL}_N$ and $\rho': \mathrm{GL}_n \rightarrow \mathrm{GL}_N$ define the same representation in different bases. Then there is a change of basis matrix S such that $\rho'(g) = S\rho(g)S^{-1}$. If $S = (s_{ij})_{1 \leq i, j \leq N}$ and $S^{-1} = (s^{ij})_{1 \leq i, j \leq N}$, this means the entries of $\rho'(g)$ are given by

$$\rho'(g)_{kl} = \sum_{a,b=1}^N s_{ka} \rho(g)_{ab} s^{bl},$$

1

making it clear that if ρ is polynomial, then so is ρ' (and symmetrically, if ρ' is, then ρ is). Therefore ρ' is polynomial if and only if ρ is, and whether a representation is polynomial is independent of the basis of V .

Example 2.2. The following give examples and a non-example of polynomial representations.

- (i) The *defining representation* $\rho: \mathrm{GL}_n \rightarrow \mathrm{GL}_n$, $g \mapsto g$ is trivially a polynomial representation since ρ is the identity map.
- (ii) The determinant $\det: \mathrm{GL}_n \rightarrow \mathbb{C}^\times = \mathrm{GL}_1$ is a representation since $\det(g_1 g_2) = \det(g_1) \det(g_2)$ for all $g_1, g_2 \in \mathrm{GL}_n$. This is in fact a polynomial representation since $\det g$ is a polynomial in the entries of g .
- (iii) One may check that $\rho: \mathrm{GL}_2 \rightarrow \mathrm{GL}_3$ defined by

$$\rho \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} g_{11}^2 & g_{11}g_{12} & g_{12}^2 \\ 2g_{11}g_{21} & g_{11}g_{22} + g_{12}g_{21} & 2g_{12}g_{22} \\ g_{21}^2 & g_{21}g_{22} & g_{22}^2 \end{pmatrix}$$

is a representation. Once one knows it is a representation, it is indeed polynomial. As we will see, this is the polynomial GL_2 -module associated to the partition $\lambda = (2)$. After developing the theory of these modules, we won't have to do the tedious computation to check this is a representation.

- (iv) One may check that $\rho: \mathrm{GL}_n \rightarrow \mathrm{GL}_2$ defined by

$$\rho(g) = \begin{pmatrix} 1 & \log(\det g) \\ 0 & 1 \end{pmatrix}$$

is a representation. This is *not* polynomial though since $\rho(g)_{12} = \log(\det g)$ is not a polynomial in the entries of g .

Every polynomial representation of GL_n is semisimple ([5, §1]), so classifying all polynomial representations of GL_n amounts to classifying all irreducible polynomial representations. Our goal throughout the rest of this report is to construct all such irreducible representations and investigate their structure.

3. MOTIVATING THE DEFINITION OF THE D^λ MODULES

In this section, we will consider a natural action of GL_n on a polynomial ring. Our exposition is largely inspired by [6], which captures some of the key ideas of [2], translated into modern mathematical language. This action will motivate the definition of the irreducible polynomial modules we give in §5.

Consider the polynomial ring in n^2 variables $\{z_{ij}\}_{1 \leq i, j \leq n}$, namely

$$V = \mathbb{C}[z_{11}, z_{12}, \dots, z_{nn}] = \mathbb{C} \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{bmatrix}.$$

Our second notation suggests we can think of this ring as the vector space of polynomial functions on the space of $n \times n$ matrices. Considering $f \in V$ as a function in this way, define for $A \in \mathrm{M}_n(\mathbb{C})$ and $g \in \mathrm{GL}_n$,

$$(g \cdot f)(A) = f(Ag).$$

This turns V into a GL_n -module since for $g, h \in \mathrm{GL}_n$ and $f \in V$,

$$((gh) \cdot f)(A) = f(A(gh)) = f((Ag)h) = (h \cdot f)(Ag) = (g \cdot (h \cdot f))(A),$$

and for $f_1, f_2 \in V$,

$$(g \cdot (f_1 + f_2))(A) = (f_1 + f_2)(Ag) = f_1(Ag) + f_2(Ag) = (g \cdot f_1)(A) + (g \cdot f_2)(A) = (g \cdot f_1 + g \cdot f_2)(A).$$

Observe the definition of this action means, in particular, that for $g = (g_{ij})_{1 \leq i, j \leq n} \in GL_n$ and $z_{ab} \in V$,

$$g \cdot z_{ab} = \sum_{k=1}^n z_{ak} g_{kb}.$$

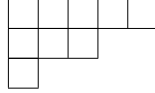
Note that $g \cdot z_{ab}$ has degree one. Furthermore, the action is multiplicative, since

$$(g \cdot (f_1 f_2))(A) = (f_1 f_2)(Ag) = f_1(Ag) f_2(Ag) = (g \cdot f_1)(A) (g \cdot f_2)(A) = ((g \cdot f_1)(g \cdot f_2))(A).$$

These two observations together imply that the action of $g \in GL_n$ does not change the degree of the polynomial it acts on. This tells us that the polynomials of any fixed degree are a GL_n -invariant subspace of V . As such, V is far from being an irreducible module. As we want to find the irreducible polynomial representations of GL_n , this leads to the definition of the D^λ modules, which are irreducible.

4. COMBINATORIAL PRELIMINARIES

We recall some basic facts about Young tableaux to fix notation and conventions before defining the D^λ modules. A *partition* of n is a weakly decreasing sequence of non-negative integers that sum to n . The number of *parts* of a partition is the number of strictly positive integers in the sequence (also known as the *length*). A partition can be identified with its *Young diagram*, a left justified set of boxes with λ_i boxes in row i . For example, the partition $\lambda = (5, 3, 1)$ of 9 has the following Young diagram:



A *filling* T of shape λ is an assignment of positive integers to each box in the Young diagram of λ . For us specifically, when considering a GL_n -module D^λ , a *filling* T will always be considered as having entries in $[n] = \{1, 2, \dots, n\}$. A *semistandard tableau* is a filling T such that the entries are weakly increasing along each row (left to right) and strictly increasing down each column. A *standard tableau* T is a semistandard tableau such that each number in $[n]$ appears exactly once. Below, we have a semistandard tableau on the left and a standard tableau on the right, both of shape $\lambda = (5, 3, 1)$:

1	2	5	5	5
4	4	7		
6				

1	2	6	7	9
3	4	5		
8				

The reader familiar with the representation theory of the symmetric group will recall that the irreducible S_n -modules have basis elements corresponding to standard tableaux. As a vector space, the D^λ modules are spanned by elements corresponding to fillings of shape λ . It turns out, basis elements of this module correspond to semistandard tableaux.

5. DEFINING THE D^λ MODULES

Let λ be a partition with at most n parts. We will define a GL_n -module D^λ that will be seen to be polynomial and irreducible. We are again inspired by [6]. In [3, §8.1], Fulton defines a more general module, called the *Schur module*, as the solution to a universal problem. However, the Schur module can also be embedded into a polynomial ring (see [3, §8.1, Lemma 3]). We will essentially take the embedding to be our definition to avoid the technicalities of Fulton's definition and to maintain the classical spirit of Deryuts's construction.

For some indices $i_1, \dots, i_p \in [n]$ such that $p \leq n$, denote by $D_{i_1, \dots, i_p}$ the minor

$$D_{i_1, \dots, i_p} = \det(z_{k, i_l})_{1 \leq k, l \leq p} = \det \begin{pmatrix} z_{1, i_1} & \cdots & z_{1, i_p} \\ \vdots & \ddots & \vdots \\ z_{p, i_1} & \cdots & z_{p, i_p} \end{pmatrix} \in V.$$

Note that $D_{i_1, \dots, i_p}$ is alternating in the indices $i_1, \dots, i_p$ (meaning $D_{i_1, \dots, i_p} = 0$ if two indices i_k are the same, and $D_{i_{\sigma(1)}, \dots, i_{\sigma(p)}} = \text{sgn}(\sigma) D_{i_1, \dots, i_p}$ for all $\sigma \in S_p$), since the determinant is alternating in the columns of a matrix. Then for a filling T of shape λ , we define the polynomial $e_T \in V$ by

$$e_T = \prod_{i=1}^{\lambda_1} D_{c_i},$$

where the product is over the columns of T and c_i is the entries of the i^{th} column of T . Specifically,

$$D_{c_i} = D_{T(1, i), \dots, T(\mu_i, i)},$$

where μ_i is the number of boxes in the i^{th} column and $T(a, b)$ is the entry in the a^{th} row of the b^{th} column of T . For example, if $\lambda = (3, 2)$ and T is the filling

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 2 \\ \hline 2 & 2 & \\ \hline \end{array}$$

then

$$e_T = D_{1,2} D_{3,2} D_2 = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} \cdot \begin{vmatrix} z_{13} & z_{12} \\ z_{23} & z_{22} \end{vmatrix} \cdot z_{12} = (z_{11}z_{22} - z_{12}z_{21})(z_{13}z_{22} - z_{12}z_{23})z_{12}.$$

Note that if $p > n$, $D_{i_1, \dots, i_p}$ is not defined. This is where the requirement that λ has at most n parts is needed; otherwise for a filling T , there will be a $D_{i_1, \dots, i_p}$ term with $p > n$ in the expression for e_T .

Definition 5.1. Let λ be a partition with at most n parts. The GL_n -module D^λ is defined by

$$D^\lambda = \text{Span}_{\mathbb{C}}\{e_T : T \text{ is a filling of the shape } \lambda\} \subseteq V.$$

The action is the same as the one discussed in §3, namely that for $g \in \text{GL}_n$, $f \in D^\lambda$ and $A \in \text{M}_n(\mathbb{C})$,

$$(g \cdot f)(A) = f(Ag).$$

We know from the discussion in §3 that this satisfies the axioms of a module, provided the action is closed on D^λ , which is not immediately clear. To see this, we need the following.

Lemma 5.2. For $g = (g_{ij})_{1 \leq i, j \leq n} \in \text{GL}_n$ and indices $i_1, \dots, i_p \in [n]$,

$$g \cdot D_{i_1, \dots, i_p} = \sum_{k_1, \dots, k_p=1}^n g_{k_1, i_1} \cdots g_{k_p, i_p} D_{k_1, \dots, k_p}.$$

Proof. Note that this result is the content of Exercise 3 in [3, §8.1]. As indicated by Fulton's hint, the result is a straightforward application of the multilinearity of the determinant in the columns of a matrix. By definition of the action and using the explicit formula for the action on monomials,

$$g \cdot D_{i_1, \dots, i_p} = \det \begin{pmatrix} g \cdot z_{1, i_1} & \cdots & g \cdot z_{1, i_p} \\ \vdots & \ddots & \vdots \\ g \cdot z_{p, i_1} & \cdots & g \cdot z_{p, i_p} \end{pmatrix} = \det \begin{pmatrix} \sum_{k_1=1}^n z_{1, k_1} g_{k_1, i_1} & \cdots & \sum_{k_p=1}^n z_{1, k_p} g_{k_p, i_p} \\ \vdots & \ddots & \vdots \\ \sum_{k_1=1}^n z_{p, k_1} g_{k_1, i_1} & \cdots & \sum_{k_p=1}^n z_{p, k_p} g_{k_p, i_p} \end{pmatrix}.$$

Writing the columns of the matrix as sums of column vectors, this is

$$\det \left(\sum_{k_1=1}^n g_{k_1, i_1} \begin{pmatrix} z_{1, k_1} \\ \vdots \\ z_{p, k_1} \end{pmatrix} \cdots \sum_{k_p=1}^n g_{k_p, i_p} \begin{pmatrix} z_{1, k_p} \\ \vdots \\ z_{p, k_p} \end{pmatrix} \right),$$

or after using the multilinearity of the determinant in the columns of the matrix,

$$\sum_{k_1, \dots, k_p=1}^n g_{k_1, i_1} \cdots g_{k_p, i_p} \det \begin{pmatrix} z_{1, k_1} & \cdots & z_{1, k_p} \\ \vdots & \ddots & \vdots \\ z_{p, k_1} & \cdots & z_{p, k_p} \end{pmatrix} = \sum_{k_1, \dots, k_p=1}^n g_{k_1, i_1} \cdots g_{k_p, i_p} D_{k_1, \dots, k_p}.$$

□

We are now in a position to see that the action is closed.

Proposition 5.3. *The action of GL_n on D^λ is closed, and hence D^λ is a GL_n -module.*

Proof. Let T be a filling of λ with entries $T(a, b)$. Using the definition of e_T and Lemma 5.2,

$$\begin{aligned} g \cdot e_T &= g \cdot \left(\prod_{i=1}^{\lambda_1} D_{c_i} \right) = \prod_{i=1}^{\lambda_1} g \cdot D_{T(1, i), \dots, T(\mu_i, i)} \\ &= \prod_{i=1}^{\lambda_1} \sum_{k_{1, i}, \dots, k_{\mu_i, i}=1}^n g_{k_{1, i}, T(1, i)} \cdots g_{k_{\mu_i, i}, T(\mu_i, i)} D_{k_{1, i}, \dots, k_{\mu_i, i}}. \end{aligned}$$

This expands to a linear combination of terms of the form

$$\prod_{i=1}^{\lambda_1} D_{k_{1, i}, \dots, k_{\mu_i, i}},$$

but this is exactly $e_{T'}$ if T' is a filling with entries given by $T'(a, b) = k_{a, b}$. So $g \cdot e_T \in D^\lambda$ as it is a linear combination of $e_{T'}$ terms. □

In Definition 5.1, we took the linear span of the e_T polynomials where T is any filling of shape λ . However, these polynomials are not all linearly independent. The following theorem tells us it suffices to restrict to semistandard tableaux T to form a basis of our module.

Theorem 5.4. *The module D^λ has a basis*

$$\{e_T : T \text{ is a semistandard tableau of shape } \lambda \text{ with entries in } [n]\}.$$

Proof. Regrettably, we do not have sufficient space in this report to prove this. The interested reader can consult [3, §8.1, Theorem 1] and the following corollary for the proof. □

After identifying this basis, one sees as a corollary to the proof of Proposition 5.3 that the D^λ modules are polynomial. When T is semistandard,

$$g \cdot e_T = \prod_{i=1}^{\lambda_1} \sum_{k_{1, i}, \dots, k_{\mu_i, i}=1}^n g_{k_{1, i}, T(1, i)} \cdots g_{k_{\mu_i, i}, T(\mu_i, i)} D_{k_{1, i}, \dots, k_{\mu_i, i}}$$

is the action on a basis vector. When this product is written as a linear combination of basis vectors, the coefficients will be polynomials in the g_{ij} entries. After collecting these polynomial coefficients into a matrix representation, D^λ can be identified as a polynomial representation.

Theorem 5.4 gives us a nice combinatorial interpretation of the dimension of the GL_n -module D^λ , namely, it is the number of semistandard Young tableaux of shape λ with entries from $[n]$. Barcelo and Ram [1, Part I, §3] provide the following interesting formula for this number:

$$\prod_{x \in \lambda} \frac{n + c(x)}{h_x},$$

where $c(x)$ is the content of the box x and h_x is the hook length at the box x (see [1, Part I, §2] to review the definition of these quantities). In the proceeding examples, the dimension will be small enough that we will be able to write out all the semistandard tableaux by inspection. However, this formula is useful for cases when such an enumeration is not straightforward.

6. EXAMPLES OF THE D^λ MODULES

Example 6.1. We give examples of the D^λ modules, seeing in particular how some well-known representations arise as D^λ modules.

- (i) Let $\lambda = (1)$. The GL_n -module $D^{(1)}$ has a basis with elements corresponding to the semistandard tableaux of shape (1) , namely $\{e_{[l]} : 1 \leq l \leq n\}$. By Lemma 5.2, the action of $g = (g_{ij})_{1 \leq i, j \leq n} \in \mathrm{GL}_n$ on $e_{[l]}$ is

$$g \cdot e_{[l]} = g \cdot D_l = \sum_{k=1}^n g_{k,l} D_k = \sum_{k=1}^n g_{k,l} e_{[k]}.$$

This is the *defining representation*.

- (ii) Let $\lambda = (1^n)$ and consider the GL_n -module $D^{(1^n)}$. There is a single semistandard Young tableau T of shape (1^n) , namely

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$$

Then $D^{(1^n)}$ is one-dimensional with basis e_T . Using Lemma 5.2, the action of $g = (g_{ij})_{1 \leq i, j \leq n} \in \mathrm{GL}_n$ on e_T is

$$g \cdot e_T = g \cdot D_{1,2,\dots,n} = \sum_{k_1,\dots,k_n=1}^n g_{k_1,1} \cdots g_{k_n,n} D_{k_1,\dots,k_n}.$$

Since $D_{k_1,\dots,k_n}$ is alternating in the indices, we have

$$g \cdot e_T = \sum_{\sigma \in S_n} g_{\sigma(1),1} \cdots g_{\sigma(n),n} \mathrm{sgn}(\sigma) D_{1,\dots,n} = \det(g^t) D_{1,\dots,n} = \det(g) e_T,$$

where we have used the Leibniz formula for the determinant. Hence, $D^{(1^n)}$ is the *determinant representation* of GL_n .

- (iii) Let $\lambda = (2)$. We will calculate the matrix representation of the GL_2 -module $D^{(2)}$ given in Example 2.2 (iii). Corresponding to the three semistandard tableaux of shape (2) with entries from $\{1, 2\}$, we have a basis for $D^{(2)}$ given by the polynomials

$$e_{\begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array}} = D_1 D_1 = z_{11}^2, \quad e_{\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}} = D_1 D_2 = z_{11} z_{12}, \quad e_{\begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array}} = D_2 D_2 = z_{12}^2.$$

It is easiest to calculate the action of $g = (g_{ij})_{1 \leq i, j \leq 2} \in \mathrm{GL}_2$ on the D_i then use that the action is multiplicative. Applying Lemma 5.2,

$$\begin{aligned} g \cdot D_1 &= \sum_{k_1=1}^2 g_{k_1,1} D_{k_1} = g_{11} D_1 + g_{21} D_2, \\ g \cdot D_2 &= \sum_{k_1=1}^2 g_{k_1,2} D_{k_1} = g_{12} D_1 + g_{22} D_2. \end{aligned}$$

So then,

$$\begin{aligned} g \cdot e_{\boxed{11}} &= (g \cdot D_1)(g \cdot D_1) \\ &= g_{11}^2 D_1 D_1 + 2g_{11}g_{21} D_1 D_2 + g_{21}^2 D_2 D_2 \\ &= g_{11}^2 e_{\boxed{11}} + 2g_{11}g_{21} e_{\boxed{12}} + g_{21}^2 e_{\boxed{22}}, \\ g \cdot e_{\boxed{12}} &= (g \cdot D_1)(g \cdot D_2) \\ &= g_{11}g_{12} D_1 D_1 + (g_{11}g_{22} + g_{12}g_{21}) D_1 D_2 + g_{21}g_{22} D_2 D_2 \\ &= g_{11}g_{12} e_{\boxed{11}} + (g_{11}g_{22} + g_{12}g_{21}) e_{\boxed{12}} + g_{21}g_{22} e_{\boxed{22}}, \\ g \cdot e_{\boxed{22}} &= (g \cdot D_2)(g \cdot D_2), \\ &= g_{12}^2 D_1 D_1 + 2g_{12}g_{22} D_1 D_2 + g_{22}^2 D_2 D_2 \\ &= g_{12}^2 e_{\boxed{11}} + 2g_{12}g_{22} e_{\boxed{12}} + g_{22}^2 e_{\boxed{22}}. \end{aligned}$$

Therefore, the matrix representation $\rho: \mathrm{GL}_2 \rightarrow \mathrm{GL}(D^{(2)})$, written with respect to the basis $\{e_{\boxed{11}}, e_{\boxed{12}}, e_{\boxed{22}}\}$, is

$$\rho(g) = \begin{pmatrix} g_{11}^2 & g_{11}g_{12} & g_{12}^2 \\ 2g_{11}g_{21} & g_{11}g_{22} + g_{12}g_{21} & 2g_{12}g_{22} \\ g_{21}^2 & g_{21}g_{22} & g_{22}^2 \end{pmatrix}.$$

7. WEIGHT SPACES AND THE IRREDUCIBILITY OF THE D^λ MODULES

The theory of *weights* plays an important role in the representation theory of Lie groups (see [4, §14, §15]). In this section, we will introduce weights and weight vectors, and use some general theory to see why the D^λ modules form a complete set of irreducible polynomial modules.

Let $H \leq \mathrm{GL}_n$ denote the subgroup of diagonal matrices. An element in H will be written $x = \mathrm{diag}(x_1, \dots, x_n)$, where each $x_i \neq 0$.

Definition 7.1. Let V be a representation of GL_n . A vector $v \in V$ is said to be a *weight vector* with *weight* $\alpha = (\alpha_1, \dots, \alpha_n)$, where each $\alpha_i \in \mathbb{Z}$, if

$$x \cdot v = x^\alpha v$$

for all $x \in H$. The notation x^α means $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Recall that a filling T of the shape λ is said to have *weight* or *content* $\alpha = (\alpha_1, \dots, \alpha_n)$ if 1 appears α_1 times in T , 2 appears α_2 times in T , and so on. The weight of T is denoted $w(T)$. Naturally, the weight of a filling T corresponds to the weight of e_T as a weight vector in D^λ .

Proposition 7.2 ([3, §8.2]). *Let D^λ be a representation of GL_n . For any filling T , e_T is a weight vector with weight $w(T)$.*

Proof. Let $x = (x_{ij})_{1 \leq i, j \leq n} \in H$ such that $x_{ij} = 0$ for $i \neq j$ (we write x in this way, instead of $\text{diag}(x_1, \dots, x_n)$, to use Lemma 5.2). Applying Lemma 5.2,

$$\begin{aligned} x \cdot D_{T(1,i), \dots, T(\mu_i, i)} &= \sum_{k_1, \dots, k_{\mu_i}=1}^n x_{k_1, T(1,i)} \cdots x_{k_{\mu_i}, T(\mu_i, i)} D_{k_1, \dots, k_{\mu_i}} \\ &= x_{T(1,i), T(1,i)} \cdots x_{T(\mu_i, i), T(\mu_i, i)} D_{T(1,i), \dots, T(\mu_i, i)}. \end{aligned}$$

If $w(T) = (\alpha_1, \dots, \alpha_n)$, then

$$\begin{aligned} x \cdot e_T &= \prod_{i=1}^{\lambda_1} x \cdot D_{T(1,i), \dots, T(\mu_i, i)} \\ &= \prod_{i=1}^{\lambda_1} x_{T(1,i), T(1,i)} \cdots x_{T(\mu_i, i), T(\mu_i, i)} D_{T(1,i), \dots, T(\mu_i, i)} \\ &= x_{11}^{\alpha_1} \cdots x_{nn}^{\alpha_n} e_T \\ &= x^{w(T)} e_T. \end{aligned}$$

□

For a weight $\alpha = (\alpha_1, \dots, \alpha_n)$, there is the corresponding *weight space*,

$$V_\alpha = \{v \in V : x \cdot v = x^\alpha v \text{ for all } x \in H\}.$$

Since the set of e_T with T semistandard form a basis for D^λ , and in light of Proposition 7.2, D^λ has a basis of weight vectors (see [3, §8.2]). This gives rise to the *weight space decomposition*,

$$D^\lambda = \bigoplus_{\alpha} V_{\alpha},$$

where α runs over weights of the form $w(T)$ with T semistandard of shape λ . Note that this is a direct sum as a vector space, not as a module. The dimension of V_α is the number of semistandard Young tableaux of shape λ and weight α . This quantity is called the *Kostka number* $K_{\lambda\alpha}$.

Example 7.3. Let $\lambda = (2, 1)$ and consider the GL_3 -module $D^{(2,1)}$. The following list gives the semistandard tableaux that correspond to basis elements, along with their weights:

$$\begin{array}{cccccccc} \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array} \\ (2, 1, 0) & (2, 0, 1) & (1, 2, 0) & (1, 1, 1) & (1, 1, 1) & (1, 0, 2) & (0, 2, 1) & (0, 1, 2) \end{array}$$

Since $e_{\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}}$ is a weight vector of weight $w\left(\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}\right) = (2, 1, 0)$, we know for example that

$$\text{diag}(x_1, x_2, x_3) \cdot e_{\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}} = x_1^2 x_2^1 x_3^0 e_{\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}}.$$

Reading off the weights of all the semistandard tableaux above, we have the weight space decomposition

$$D^{(2,1)} = V_{(2,1,0)} \oplus V_{(2,0,1)} \oplus V_{(1,2,0)} \oplus V_{(1,1,1)} \oplus V_{(1,0,2)} \oplus V_{(0,2,1)} \oplus V_{(0,1,2)}.$$

Here, all the weight spaces are one-dimensional except $V_{(1,1,1)}$, which is two-dimensional since there are two semistandard tableaux of weight $(1, 1, 1)$.

We now briefly explain why the D^λ modules form a complete set of irreducible polynomial representations. Further details can be found in [3, §8.2], while the complete theory is developed in [4, §14, §15]. Let $B \leq \mathrm{GL}_n$ denote the so-called Borel subgroup of upper triangular matrices. A weight vector $v \in V$ is said to be a *highest weight vector* if $B \cdot v = \mathbb{C}^\times v$. The theory of weights tells us that a complex representation of GL_n is irreducible if and only if the representation has a unique (up to a scalar multiple) highest weight vector. It turns out that the module D^λ has such a highest weight vector, namely e_U , where U is the semistandard tableaux with i in every box of the i^{th} row. As such, the D^λ modules are irreducible. Furthermore, two representations are isomorphic if and only if their highest weight vectors have the same weight. For a polynomial representation, the possible highest weights α are those such that $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 0$, meaning α is a partition with at most n parts. Observe that the highest weight vector of D^λ , e_U , has weight $w(U) = \lambda$. So for any highest weight λ , D^λ is a module with a highest weight vector of weight λ , and thus these modules form a complete set of irreducible polynomial representations.

8. CHARACTERS OF THE D^λ MODULES

Like in the representation theory of finite groups, characters play an important role in the representation theory of Lie groups. In this section, we define and compute the characters of the D^λ modules and see an interesting application.

If $x = \mathrm{diag}(x_1, \dots, x_n) \in \mathrm{GL}_n$, and $\rho: \mathrm{GL}_n \rightarrow V$ is a representation, the character is

$$\chi_V(x) = \mathrm{Tr}(\rho(x)).$$

Note that this means the character is only defined on diagonal matrices.

To compute the character of the GL_n -module D^λ , we can use the fact that the module has a basis of weight vectors. Indeed, if $e_T \in D^\lambda$ such that T is semistandard, we have for $x = \mathrm{diag}(x_1, \dots, x_n)$ that

$$x \cdot e_T = x^{w(T)} e_T.$$

So the action of x is diagonal as a matrix representation, and the trace is

$$\mathrm{Tr}(\rho(g)) = \sum_T x^{w(T)},$$

where the sum is over all semistandard Young tableaux of shape λ . These polynomials are called *Schur polynomials* and are denoted by $s_\lambda(x_1, \dots, x_n)$. The Schur polynomials have many interesting combinatorial properties in their own right; for example, they are symmetric polynomials, meaning $s_\lambda(x_1, \dots, x_n) = s_\lambda(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for all $\sigma \in S_n$. In fact, the Schur polynomials form an additive basis for the ring of symmetric polynomials (for further details on Schur polynomials, including proofs of these facts, see [3, §6]).

Two modules D^λ are isomorphic if and only if they have the same character. We have the following theorem that offers a new perspective on the classification of the irreducible polynomial GL_n -modules using characters.

Theorem 8.1 ([5, Theorem 1.2]). *Let λ be a partition with at most n parts. The Schur function $s_\lambda(x_1, \dots, x_n)$ is an irreducible polynomial character of GL_n , different λ yield different characters, and every irreducible polynomial character has this form.*

Example 8.2. The character of the GL_3 -module $D^{(2,1)}$, by reading off the list of semistandard Young tableaux and their weights given in Example 7.3, is

$$\begin{aligned} s_{(2,1)}(x_1, \dots, x_n) &= \sum_T x^{w(T)} \\ &= x_1^2 x_2^1 x_3^0 + x_1^2 x_2^0 x_3^1 + x_1^1 x_2^2 x_3^0 + x_1^1 x_2^1 x_3^1 \\ &\quad + x_1^1 x_2^1 x_3^1 + x_1^1 x_2^0 x_3^2 + x_1^0 x_2^2 x_3^1 + x_1^0 x_2^1 x_3^2 \\ &= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2. \end{aligned}$$

Since the character of a direct sum of representations is the sum of characters (that is, $\chi_{V \oplus W} = \chi_V + \chi_W$), the problem of finding the multiplicities of irreducibles in any representation is equivalent to expanding the character in terms of the Schur polynomials (see [5, §1]). This leads to a rich interplay between combinatorics and representation theory; see the survey article [1] for an overview of the subject of combinatorial representation theory.

Example 8.3. We give a simple but illustrative example of how characters can be used to decompose an arbitrary representation of GL_n into irreducibles. Let GL_n act on the space of $n \times n$ matrices $V = M_n(\mathbb{C})$ by the formula

$$g \cdot A = gAg^t.$$

Let $\rho: \mathrm{GL}_n \rightarrow \mathrm{GL}(V)$ denote the corresponding matrix representation. To calculate the character of this representation, take $x = \mathrm{diag}(x_1, \dots, x_n) \in \mathrm{GL}_n$ and consider the action on a basis vector E_{ij} . We get,

$$x \cdot E_{ij} = xE_{ij}x^t = xE_{ij}x = x_i x_j E_{ij}.$$

This means that the matrix representation $\rho(x)$ is diagonal with entries $x_i x_j$ for $i, j \in [n]$. The character is

$$\begin{aligned} \chi_V(x_1, \dots, x_n) &= \mathrm{Tr}(\rho(x)) \\ &= \sum_{i,j=1}^n x_i x_j \\ &= \sum_{1 \leq i \leq j \leq n} x_i x_j + \sum_{1 \leq j < i \leq n} x_i x_j \\ &= \sum_{T \in \mathrm{SSYT}((2))} x^{w(T)} + \sum_{T \in \mathrm{SSYT}((1,1))} x^{w(T)} \\ &= s_{(2)}(x_1, \dots, x_n) + s_{(1,1)}(x_1, \dots, x_n). \end{aligned}$$

Here $\mathrm{SSYT}(\lambda)$ is the set of semistandard Young tableaux of shape λ . This expression for the character χ_V tells us that

$$V \cong D^{(2)} \oplus D^{(1,1)}.$$

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