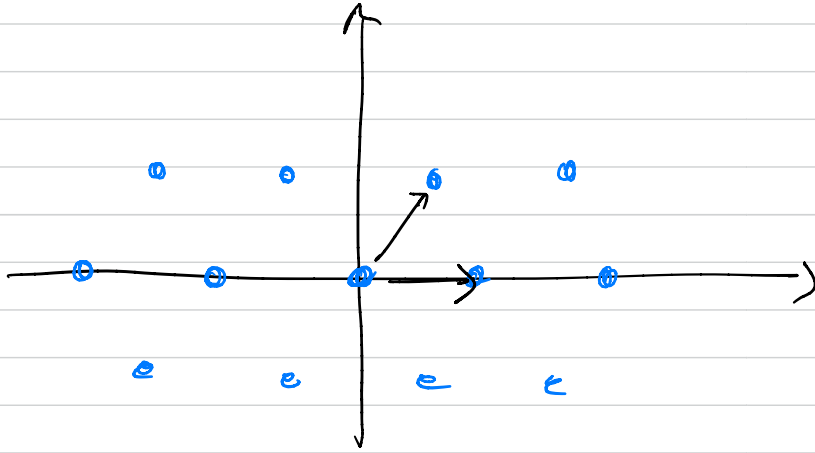


Modular forms talk 1

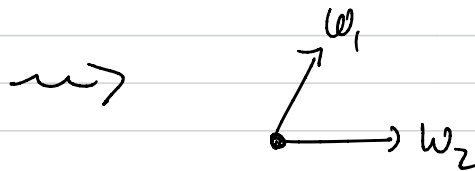
Def: a group $\Gamma \leq \mathbb{C}$ is a lattice if there exists a $\mathbb{Z}$ -basis of Γ that is an $\mathbb{R}$ -basis of $\mathbb{C}$.



We're interested in $\mathcal{R}$, the set of all lattices.

$$M := \left\{ (\omega_1, \omega_2) \in \mathbb{C}^x \times \mathbb{C}^x : \operatorname{Im}\left(\frac{\omega_1}{\omega_2}\right) > 0 \right\}$$

$$\left(\text{Note: } \omega_1 = \left(\frac{\omega_1}{\omega_2}\right) \cdot \omega_2, \operatorname{Im}\left(\frac{\omega_1}{\omega_2}\right) \geq 0 \right)$$



We have a map $M \rightarrow \mathcal{R}$,

$$(\omega_1, \omega_2) \mapsto \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 =: \Gamma(\omega_1, \omega_2)$$

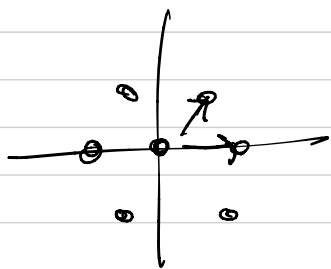
that is surj.

$$\mathbb{C}^x \curvearrowright \mathcal{R} \quad (\mathbb{C}^x \curvearrowright M) \quad \text{by}$$

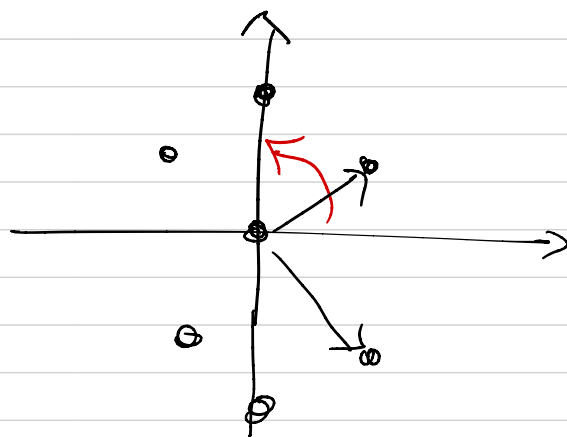
$$\lambda \cdot \Gamma = \lambda \Gamma \quad (\lambda \cdot (\omega_1, \omega_2) = (\lambda\omega_1, \lambda\omega_2))$$

Example:

$$2e^{i\frac{\pi}{6}}$$

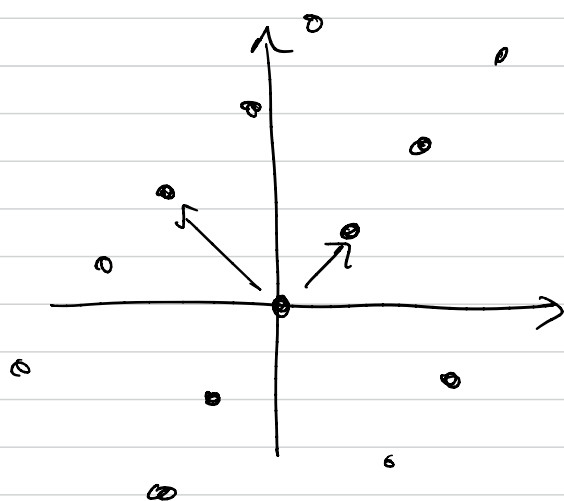


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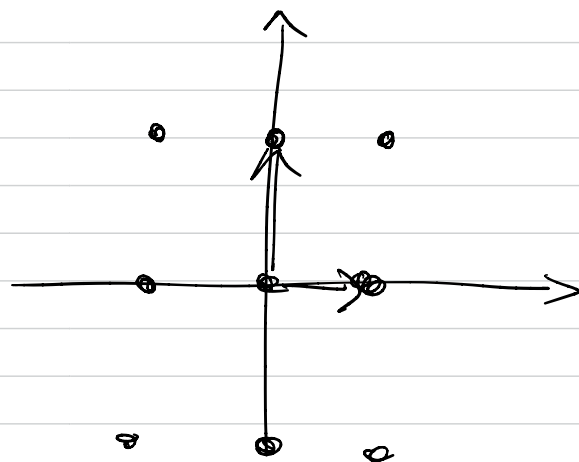


We can remove redundancy by normalising:

$$\Gamma(\omega_1, \omega_2) \mapsto \frac{1}{\omega_2} \cdot \Gamma(\omega_1, \omega_2) = \Gamma\left(\frac{\omega_1}{\omega_2}, 1\right).$$



~>



Thus, $\mathcal{M}/\mathbb{C}^\times \xrightarrow{\sim} H := \{z \in \mathbb{C} : \text{Im}(z) \geq 0\},$

$$\Gamma(\omega_1, \omega_2) \mapsto \Gamma\left(\frac{\omega_1}{\omega_2}, 1\right) \mapsto \frac{\omega_1}{\omega_2} \in H.$$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a\omega_1 + b\omega_2 \\ c\omega_1 + d\omega_2 \end{pmatrix}$$

is also a $\mathbb{Z}$ -basis of $\Gamma(\omega_1, \omega_2).$

If $z := \frac{\omega_1}{\omega_2}$ and $z' := \frac{\omega_1'}{\omega_2'}$, then

$$z' = \frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2} = \frac{a\frac{\omega_1}{\omega_2} + b}{c\frac{\omega_1}{\omega_2} + d} = \frac{az + b}{cz + d} =: g \cdot z$$

Lemma: $\operatorname{Im}(g \cdot z) = \frac{\operatorname{Im}(z)}{|cz + d|^2}$.

Proof:

$$\begin{aligned} g \cdot z &= \frac{az + b}{cz + d} = \frac{(az + b)(c\bar{z} + d)}{|cz + d|^2} \\ &= \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz + d|^2} \\ &= \frac{(ac|z|^2 + (ad + bc)\operatorname{Re}(z) + bd) + \overbrace{(ad - bc)\operatorname{Im}(z)i}^{=1}i}{|cz + d|^2} \end{aligned}$$

□

Prop: For two elements of M to define the same lattice it is necessary and sufficient that they are cong. mod. $SL_2(\mathbb{Z})$.

The map

$$M/\mathbb{C}^\times \longrightarrow \mathbb{H}, \quad (\omega_1, \omega_2) \mapsto \frac{\omega_1}{\omega_2}$$

transforms the action of $SL_2(\mathbb{Z})$ on

M into

$$g \cdot z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

in H .

Prop: The map $(w_1, w_2) \mapsto \frac{w_1}{w_2}$ gives, by passing to the quotient, a bijection of $\mathbb{R}/\mathbb{C}^\times$ onto H/Γ .

§1 The modular group

Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot z = \frac{-1 \cdot z + 0}{0 + -1} = z$ ($-\mathrm{Id}$ acts

trivially), we consider the modular group

$$\Gamma := \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) / \{\pm 1\}$$

to be acting on H .

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

1.2 Fundamental domain

Denote $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and note

$$S \cdot z = -\frac{1}{z}, \quad T \cdot z = z + 1$$

$$S^2 = 1, \quad (ST)^3 = 1.$$

Define $D \subseteq H$ to be the set of all $z \in H$ such that $|z| \geq 1$ and $-\frac{1}{2} \leq \operatorname{Re}(z) \leq \frac{1}{2}$.

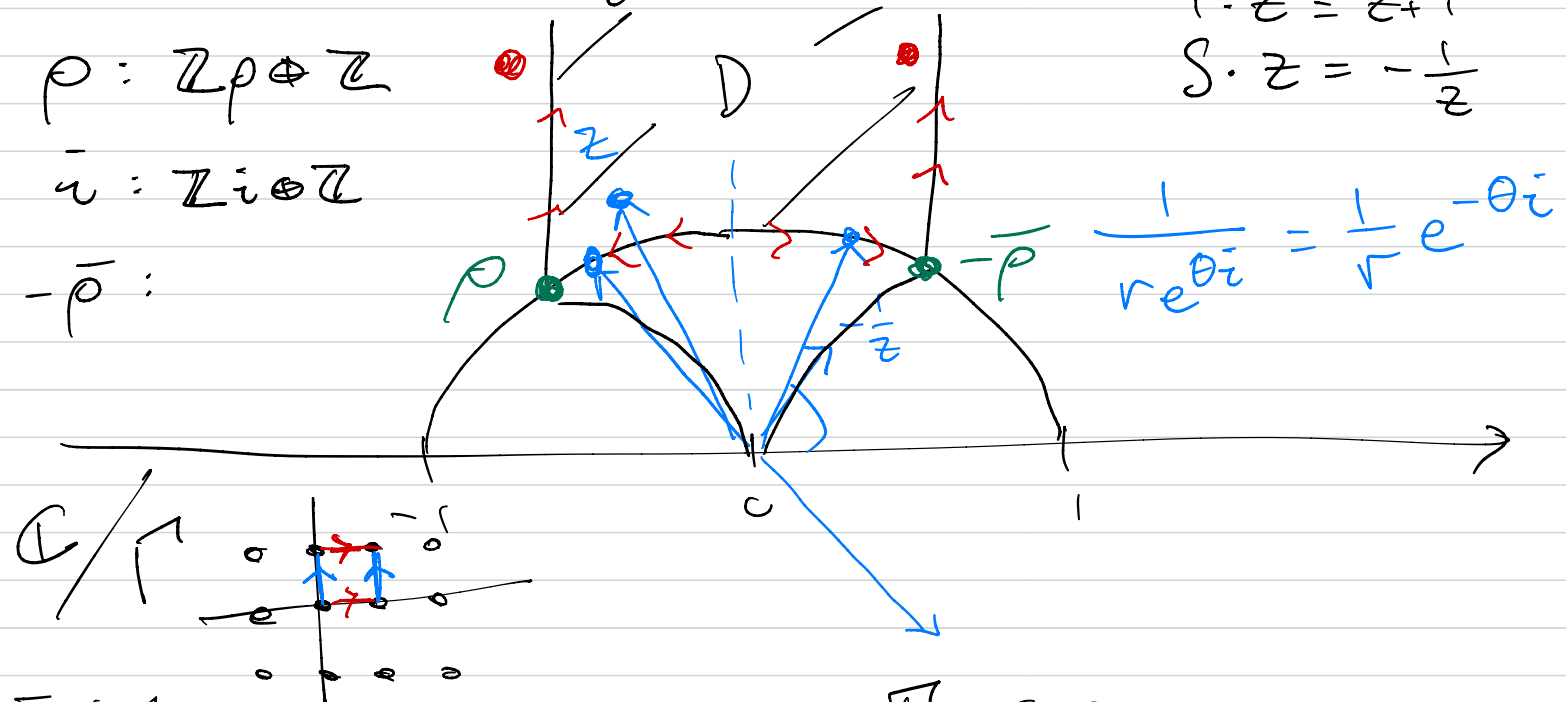
$$\rho: \mathbb{Z} \oplus \mathbb{Z}$$

$$\bar{i}: \mathbb{Z} \oplus \mathbb{Z}$$

$$-\bar{\rho}:$$

$$\begin{aligned} T \cdot z &= z + 1 \\ S \cdot z &= -\frac{1}{z} \end{aligned}$$

$$\frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$$



Intuition: $z \in D$ represents $\Gamma = \mathbb{Z}z \oplus \mathbb{Z}$.

This equals $\mathbb{Z}(z+1) \oplus \mathbb{Z}$, and is homothetic to

$$\mathbb{Z} \frac{-1}{z} \oplus \mathbb{Z} = -\frac{1}{z} (\mathbb{Z} \oplus \mathbb{Z}z).$$

Theorem 1: (1) For every $z \in H$, there exists $g \in \Gamma$ such that $g \cdot z \in D$, and $z, z' \in D$

(2) Suppose $z \neq z'$ are cong. mod. Γ . Then, $\operatorname{Re}(z) = \pm \frac{1}{2}$ and $z' = z \pm 1$, or $|z| = 1$ and $z' = -\frac{1}{z}$.

(3) Let $z \in D$ and $I(z) := \{g \in \Gamma \mid g \cdot z = z\}$. Then, $I(z) = \{1\}$ except if:

$$z = i, \text{ where } I(z) = \langle S \rangle \cong \mathbb{Z}/2\mathbb{Z},$$

$z = \rho = e^{2\pi i/3}$, where $I(z) = \langle ST \rangle \cong \mathbb{Z}/3\mathbb{Z}$
 $z = -\bar{\rho} = e^{\pi i/3}$, where $I(z) = \langle TS \rangle \cong \mathbb{Z}/3\mathbb{Z}$.

Coro: The canonical map $D \rightarrow H/G$ is surj. and its restriction to the interior of D is inj.

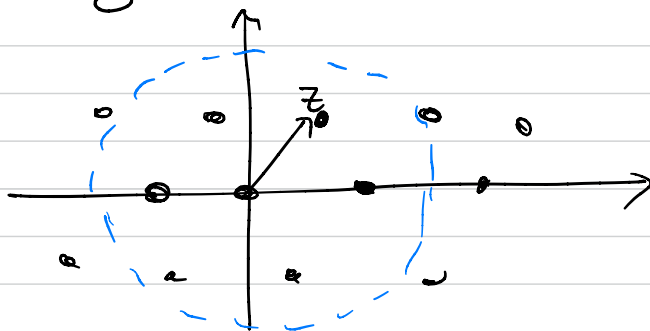
Theorem 2: G is generated by S and T .

Proof: [Thm 1 (1)] Denote $G' := \langle S, T \rangle \leq G$.
 Let $z \in H$. Want $g' \in G'$ such that $g' \cdot z \in D$.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, recall

$$\operatorname{Im}(g \cdot z) = \frac{\operatorname{Im}(z)}{|cz + d|^2}.$$

The number of pairs (c, d) such that $|cz + d|$ is less than a given number is finite

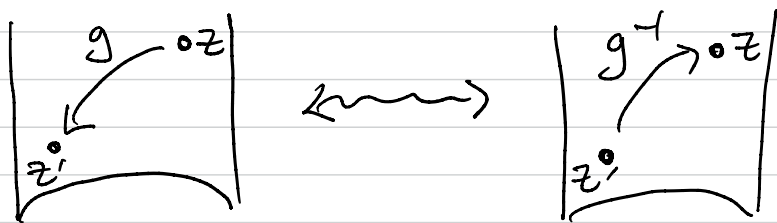


there exists $g \in G'$ such that $\operatorname{Im}(g \cdot z)$ is maximised. We can translate $g \cdot z$ left or right to get $T^n g \cdot z$ to have real part between $-\frac{1}{2}$ and $\frac{1}{2}$. We claim $z' = T_g^n \cdot z \in D$,

it suffices to check $|z'| \geq 1$. If $|z'| < 1$, the element $-\frac{1}{z'}$ would have imaginary part strictly larger than $\text{Im}(z') = \text{Im}(g \cdot z)$, a contradiction.

[Thm 1, (2) & (3)] Suppose $z' = g \cdot z$ for some $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, and $z, z' \in D$.

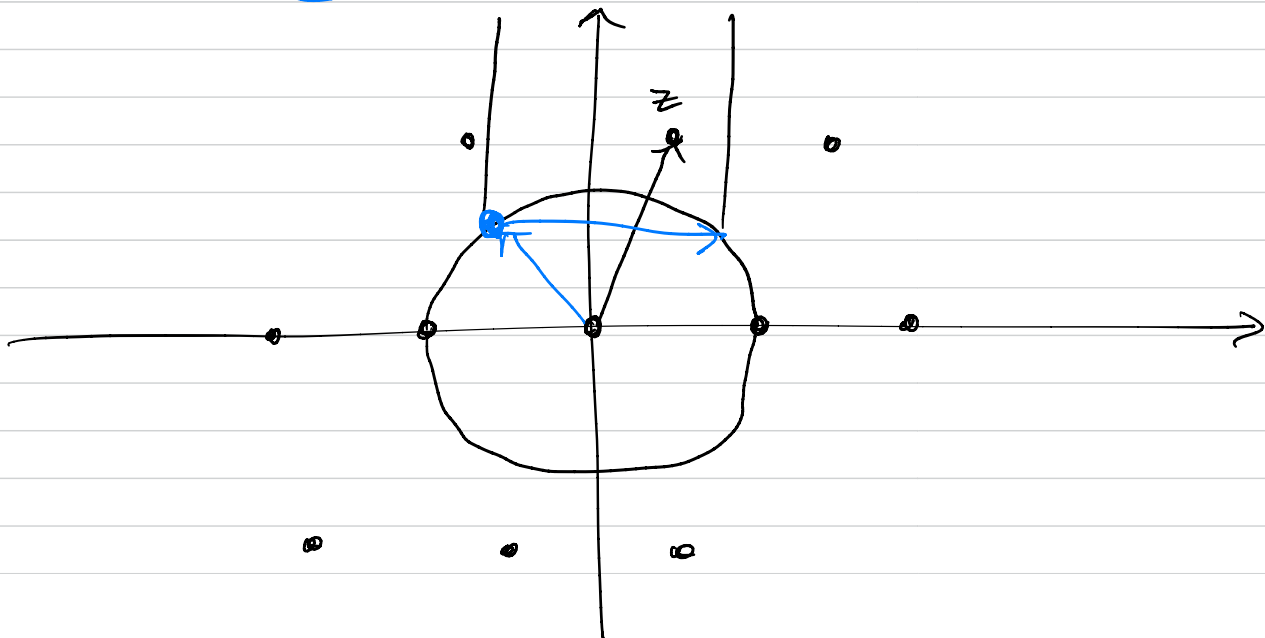
We must show either $z \neq z'$ and we're in one of the cases of (2), or $z = z'$ and g lies in one of the stabilisers in (3).



Without loss of generality, assume

$$\text{Im}(g \cdot z) = \frac{\text{Im}(z)}{|cz+d|^2} \geq \text{Im}(z)$$

Therefore, $|cz+d| \leq 1$.



Cases:

$$\textcircled{1} \begin{cases} c=0 \\ d=\pm 1 \end{cases} \quad \begin{cases} z \in D \cap S' \\ c=\pm 1 \\ d=0 \end{cases} \quad \textcircled{2} \begin{cases} z=p \\ c=\pm 1 \\ d=\pm 1 \end{cases} \quad \begin{cases} z=-\bar{p} \\ c=\pm 1 \\ d=\mp 1 \end{cases}$$

$$\textcircled{1} g = \begin{pmatrix} a & b \\ 0 & \pm 1 \end{pmatrix}, \quad ad - bc = 1 \Rightarrow \pm a = 1 \\ \Rightarrow a = d = \pm 1$$

$$z' = \frac{az+b}{cz+d} = \frac{\pm z+b}{0 \cdot z \pm 1} = z \pm b$$

If $z' \in D$, we must have $\operatorname{Re}(z) = \pm \frac{1}{2}$ and $b = \pm 1$ or $b = 0$.

$$\textcircled{2} g = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}, \quad ad - bc = 1 \Rightarrow a - b = 1.$$

$$z' = \frac{ap+b}{p+1} = \frac{a(p+1)-a+b}{p+1} = a - \frac{1}{p+1} \\ = a + p$$

$$\text{If } z = z' = p, \text{ then } a = 0 \text{ and } g = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \\ = ST.$$

$$\text{If } z' \neq z, \text{ then } a = 1 \text{ and } z' = z + 1.$$

[Thm 2] Let $g \in G$. Choose z_0 in the interior of D (say $z_0 = 2i$), and let $z := g \cdot z_0$.

We know there exists $g' \in G' = \langle S, T \rangle$ such

that $g' \cdot z \in D$. Then, z_0 and $g' \cdot z = g'g \cdot z_0$ are cong. mod. A and both lie in D . Since z_0 is in the interior, (2) and (3) imply they coincide and that $g'g = I$. Hence, $g = (g')^{-1} \in A' = \langle S, T \rangle$. $\square$

Stefano's comments (details need checking)

$$y^2 = x^3 + ax + b$$

$$\wp(z) = \frac{1}{u^2} + \sum_{\gamma \in \Gamma \setminus \{0\}} \left(\frac{1}{(u-\gamma)^2} - \frac{1}{\gamma^2} \right).$$

$\mathbb{C}/\Gamma$. If $\wp(z+w) = \wp(z)$
for all $w \in \Gamma$, $\wp$ defines
a function on $\mathbb{C}/\Gamma$.

$$\mathbb{C}/\Gamma \longrightarrow \mathbb{C}^2$$

$$z \longmapsto (\wp(z), \wp'(z)) = (x, y)$$

$$y^2 = 4x^3 - g_2x - g_3$$

Modular forms talk 2

Def: Let $k \in \mathbb{Z}$. We say f is weakly modular of weight $2k$ (or $-2k$, or k) if f is mero. and

$$(2) \quad f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z) \quad \text{for all} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

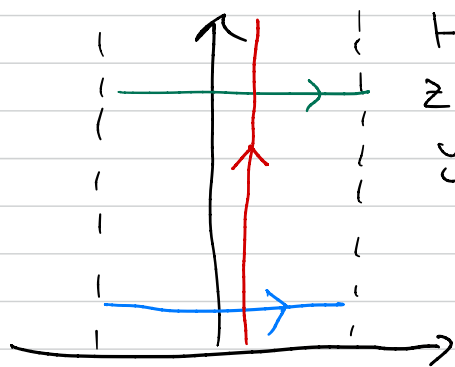
Note: (2) is equivalent to

$$(4) \quad f(z+1) = f(z), \quad \text{and}$$

$$(5) \quad f(-1/z) = z^{2k} f(z),$$

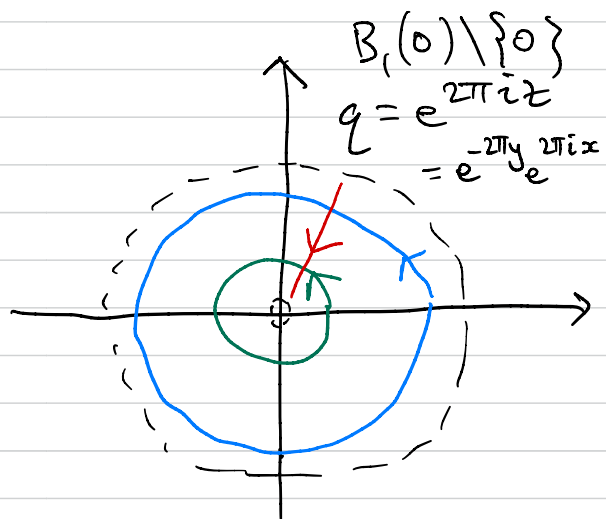
since $\Gamma = \langle S, T \rangle$.

There is a change of variables



$$H \\ z = x+yi, \\ y > 0$$

$$z \mapsto e^{\pi i z} =: q$$



The fact that $f(z+1) = f(z)$ means that f can be written as a f^n of q , which we denote $\tilde{f}(q)$. $\tilde{f}$ is mero. on the punctured disk. We say f is mero. (holo.) at ∞ if $\tilde{f}$ extends to a mero. (holo.) f^n at 0 , i.e., if on a

holo. at $0, \infty$

$$\tilde{f}(q) = \sum_{n=-\infty}^{\infty} a_n q^n$$

where $a_n = 0$ for all n suff. small ($n < 0$).

Def: A weakly modular function is called modular if it is holo. at ∞ .

If f is holo. at ∞ , $f(\infty) := \tilde{f}(0)$.

Def: A modular form is a modular f^n which holo. everywhere, incl. ∞ . If $f(\infty) = 0$, f is called a cusp form.

Therefore, a modular form of weight $2k$ is a series

$$f(z) = \sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

converging for $|q| < 1$ (i.e., $\text{Im}(z) > 0$)

verifying

$$f(-1/z) = z^{2k} f(z).$$

It is a cusp form if $a_0 = 0$.

Example: $q \prod_{n=1}^{\infty} (1 - q^n)^{24}$ is a cusp form of weight 12.

2.2 Lattice f^n s & modular f^n s

Recall

$$\mathcal{R} := \{\text{lattices in } \mathbb{C}\}, \quad M := \left\{ (w_1, w_2) \in \mathbb{C}^* \times \mathbb{C}^* : \operatorname{Im}\left(\frac{w_1}{w_2}\right) > 0 \right\}.$$

If F on $\mathcal{R}$ and $k \in \mathbb{Z}$, we say F is of weight $2k$ if

$$(7) \quad F(\lambda \Gamma) = \lambda^{-2k} F(\Gamma) \text{ for all } \Gamma \in \mathcal{R} \text{ and } \lambda \in \mathbb{C}^*.$$

Write $F(w_1, w_2)$ for the value of F on $\Gamma(w_1, w_2)$.

(7) becomes

$$(8) \quad F(\lambda w_1, \lambda w_2) = \lambda^{-2k} F(w_1, w_2).$$

F is invariant under the action of $SL_2(\mathbb{Z})$ on M .

Subbing $\lambda = \frac{1}{w_2}$,

$$(9) \quad w_2^{2k} F(w_1, w_2) = F\left(\frac{w_1}{w_2}, 1\right) =: f\left(\frac{w_1}{w_2}\right)$$

Writing $z = \frac{w_1}{w_2}$, we have

$$\begin{aligned} f(z) &= F(z, 1) = F(az+b, cz+d) \\ &= (cz+d)^{-2k} F\left(\frac{az+b}{cz+d}, 1\right) \\ &= (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \end{aligned} \tag{2}$$

Then, $(2) \Leftrightarrow (9)$, and modular f^n s of weight $2k \Leftrightarrow$ lattice f^n s of weight $2k$.

2.3 Examples of modular f^n s; Eisenstein series

Let $k > 1$, and $\Gamma \in \mathcal{R}$. Define the Eisenstein series of index k (or $2k$) by

$$(10) \quad G_k(\Gamma) := \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{\gamma^{2k}},$$

or

$$(11) \quad G_k(\omega_1, \omega_2) := \sum_{(m,n) \neq (0,0)} \frac{1}{(m\omega_1 + n\omega_2)^{2k}},$$

or

$$(12) \quad G_k(z) := \sum_{(m,n) \neq (0,0)} \frac{1}{(mz + n)^{2k}}.$$

(10) is clearly of weight $2k$. Our previous discussion implies (12) is weakly modular of weight $2k$.

Lemma 1: Let $\Gamma \in \mathcal{R}$. The series

$$\sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{|\gamma|^s}$$

is convergent for $s > 2$.

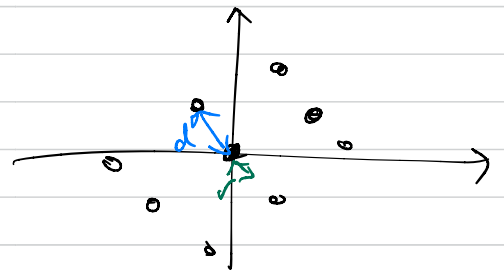
Proof: Set

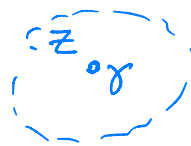
$$d := \min \{ |\gamma| : \gamma \in \Gamma \setminus \{0\} \}, \quad r := \frac{d}{2}.$$

Then the balls

$$B_r(\gamma), \quad \gamma \in \Gamma \setminus \{0\}$$

are non-overlapping.





If $z \in B_r(\gamma)$, $\gamma \neq 0$,

$$|z| = |\gamma + z - \gamma| \leq |\gamma| + r \leq |\gamma| + \frac{|\gamma|}{2} < 2|\gamma|$$

Then,

$$\frac{1}{|\gamma|} < \frac{2}{|z|} \Rightarrow \frac{1}{|\gamma|^\sigma} \leq \frac{2^\sigma}{|z|^\sigma}.$$

Integrating over $B_r(\gamma)$,

$$\frac{\pi r^2}{|\gamma|^\sigma} \leq 2^\sigma \int_{B_r(\gamma)} \frac{1}{|z|^\sigma} dA.$$

Summing over $\gamma \neq 0$,

$$\begin{aligned} \pi r^2 \sum_{\gamma \in \Gamma \setminus \{0\}} \frac{1}{|\gamma|^\sigma} &\leq 2^\sigma \int_{\bigcup_{\gamma \neq 0} B_r(\gamma)} \frac{1}{|z|^\sigma} dA \\ &\leq 2^\sigma \int_{|z| \geq r} \frac{1}{|z|^\sigma} dA. \end{aligned}$$

We just need to check the RH integral is finite:

$$\begin{aligned} \int_{|z| \geq r} \frac{1}{|z|^\sigma} dA &= \int_0^{2\pi} \int_r^\infty \frac{1}{u^\sigma} \cdot u \, du \, d\theta \\ &= 2\pi \left[\frac{u^{-\sigma+2}}{-\sigma+2} \right]_r^\infty \quad \begin{array}{l} \nearrow \text{converges iff} \\ \sigma-1 > 1 \end{array} \\ &= \frac{2\pi}{\sigma-2} \cdot \frac{1}{r^{\sigma-2}}. \end{aligned}$$

□

Prop 4: Let $k > 1$. $G_k(z)$ is a modular form of weight $2k$. We have $G_k(\infty) = 0$. (!!!)

Proof: First, consider $z \in D$:

$$|mz+n|^2 = m^2|z|^2 + 2mn\operatorname{Re}(z) + n^2 \quad \rho = e^{2\pi i/3}$$

$$\underbrace{m^2|z|^2}_{\geq 1} + \underbrace{2mn\operatorname{Re}(z)}_{\geq -\frac{1}{2}} + n^2$$

$$\geq m^2 - mn + n^2 = |m\rho + n|^2$$

Then,

$$\sum_{(m,n) \neq (0,0)} \frac{1}{|mz+n|^{2k}} \leq \sum_{(m,n) \neq (0,0)} \frac{1}{|m\rho+n|^2} \stackrel{\text{Lemma}}{<} \infty.$$

Therefore, G_k converges absolutely & uniformly on D , so G_k is holo. on D . If $z' \in gD$, $z' = g \cdot z$ for some $z \in D$ and the formula

$$G_k(z') = G_k(g \cdot z) = (cz+d)^{2k} G_k(z)$$

$$= (cg^{-1} \cdot z' + d)^{2k} G_k(g^{-1} \cdot z')$$

shows G_k is holo. on gD . Then G_k is holo. on H since the translations gD cover H .

To see G_k is holo. at ∞ , we need to show the limit as $\operatorname{Im}(z) \rightarrow \infty$ of $G_k(z)$ exists. The uniform convergence in D implies we can take the limit term by term:

$$\lim_{\operatorname{Im}(z) \rightarrow \infty} G_k(z) = \underbrace{\sum_{\substack{m \neq 0 \\ n \in \mathbb{Z}}} \lim_{\operatorname{Im}(z) \rightarrow \infty} \frac{1}{(mz+n)^{2k}}}_{=0} + \sum_{\substack{m=0 \\ n \in \mathbb{Z} \setminus \{0\}}} \lim_{\operatorname{Im}(z) \rightarrow \infty} \frac{1}{(mz+n)^{2k}}$$

$$= 2 \zeta(2k).$$

□

Relation w/ elliptic curves: If Γ is a lattice, the Weierstrass $\wp - \wp^u$ is

$$\wp_{\Gamma}(u) := \frac{1}{u^2} + \sum_{\gamma \in \Gamma \setminus \{0\}} \left(\frac{1}{(u-\gamma)^2} - \frac{1}{\gamma^2} \right).$$

If $x = \wp_{\Gamma}(u)$ and $y = \wp'_{\Gamma}(u)$, there holds

$$y^2 = 4x^3 - g_2 x - g_3, \quad \text{where } g_2 := 60 G_2, \\ g_3 := 140 G_3.$$

Therefore, $\mathbb{C}/\Gamma$ is isom. to a cubic curve.

The Eisenstein series appear in the Laurent expansion of $\wp'_{\Gamma}(u)$:

$$\wp'_{\Gamma}(u) = -\frac{1}{u^3} + \sum_{k=2}^{\infty} (2k-1) G_k(\Gamma) u^{2k-2}.$$

Moreover,

$$\Delta = g_2^3 - 27g_3^2$$



is a modular form of weight 12. One can compute $\Delta(\infty) = 0$ (requires $\wp$ values), so it is a cusp form. Δ is the discriminant of $4x^3 - g_2 x - g_3$ up to a numerical factor.