Maximum and Hitting Times for the Wiener Process

Declan Fletcher

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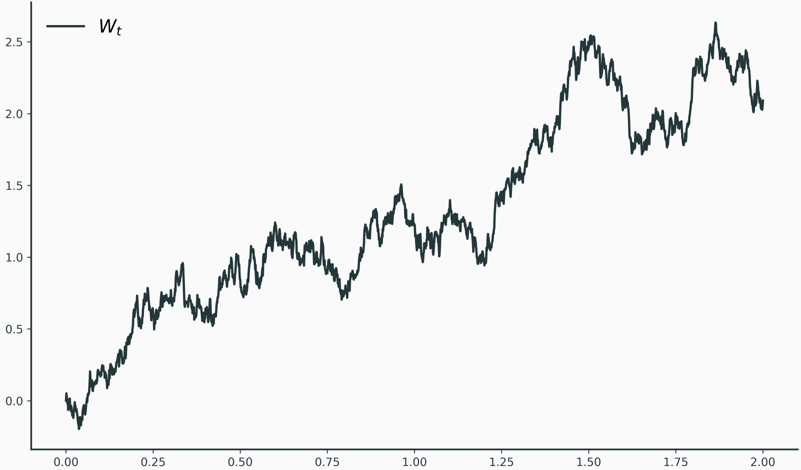
The running maximum

Let $(W_t)_{t \geq 0}$ be a Wiener process.

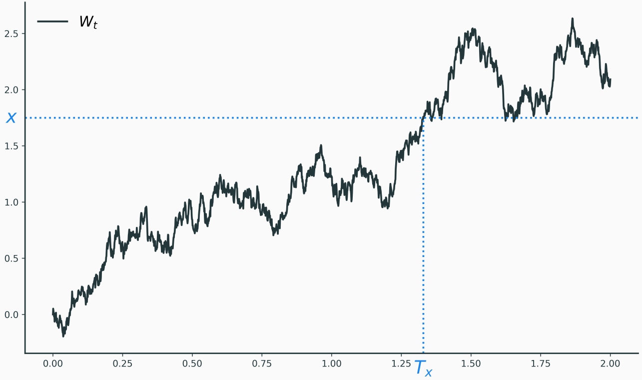
The running maximum $(M_t)_{t \geq 0}$ is $M_t := \max_{0 \leq s \leq t} W_s$.



The hitting time of (x, ∞)

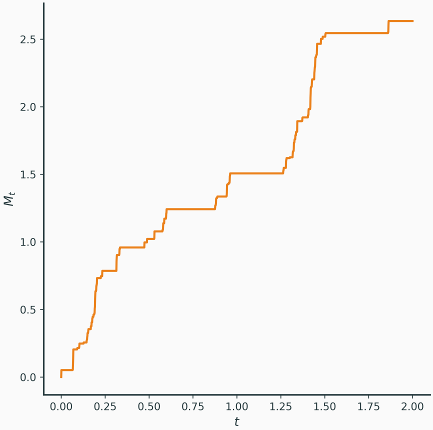
For $x \geq 0$, the hitting time T_x is the time when W_t becomes larger than x. That is,

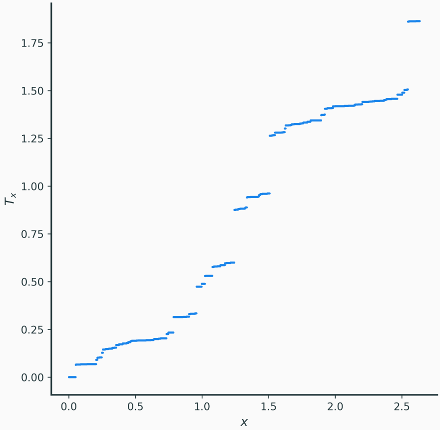
$$T_x := \inf\{t > 0 : W_t > x\}.$$



We consider $(T_x)_{x \geq 0}$ as a stochastic process.

The relationship between (M_t) and (T_x)





(1) M_t and T_x are inverse functions in the sense of Exercise 1.24.

(2) There holds $\{M_t > x\} = \{T_x < t\}$.

Pdfs of M_t and T_x

We have

$$\begin{aligned}\mathbb{P}(M_t > x) &= \mathbb{P}(T_x < t) \\ &= \mathbb{P}(T_x < t, W_t > x) + \mathbb{P}(T_x < t, W_t < x) \\ &= 2\mathbb{P}(T_x < t, W_t > x) \\ &= 2\mathbb{P}(W_t > x) \\ &= 2\left(1 - \Phi(x/\sqrt{t})\right).\end{aligned}$$

This yields the following pdfs:

$$f_{M_t}(x) = \frac{2}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \quad f_{T_x}(t) = \frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right).$$

The distribution of M_t

Fix $t \geq 0$. M_t has pdf $f_{M_t}(x) = \dfrac{2}{\sqrt{2\pi t}} \exp\left(-\dfrac{x^2}{2t}\right)$, $x \geq 0$.

This is $N(0, t)$ truncated to $[0, \infty)$.

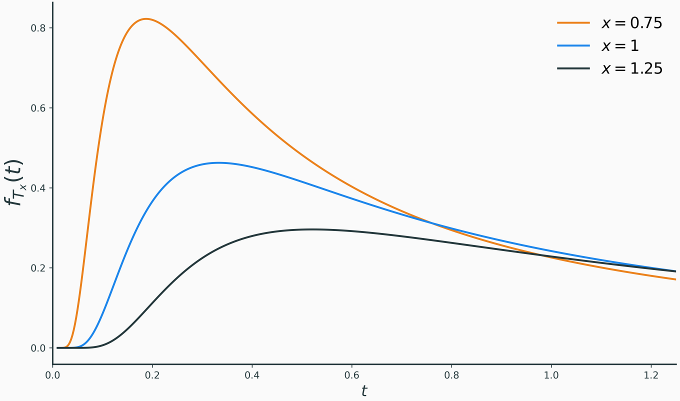
So, M_t has the same distribution as $|W_t|$.

The distribution of T_x

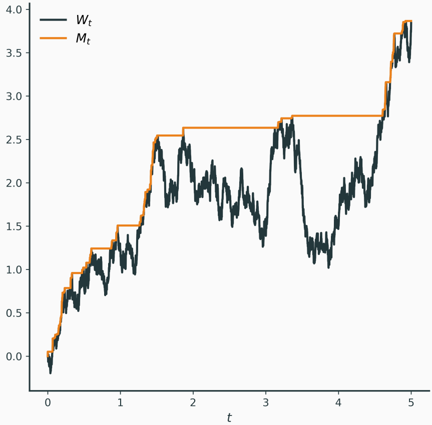
Fix $x \geq 0$. T_x has pdf $f_{T_x}(t) = \dfrac{x}{\sqrt{2\pi t^3}} \exp\left(-\dfrac{x^2}{2t}\right)$, $t \geq 0$.

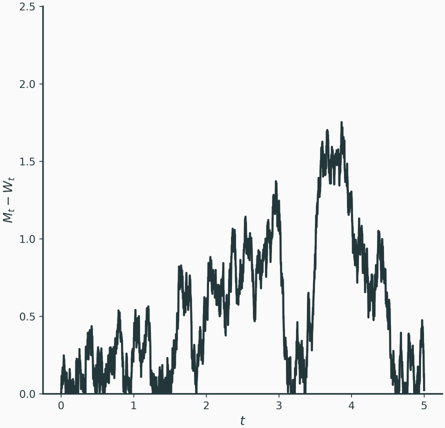
T_x has the same distribution as $\frac{x^2}{Z^2}$ if $Z \sim N(0,1)$.

It can be shown $\mathbb{P}(T_x < \infty) = 1$, but $\mathbb{E} T_x = \infty$.



The difference $M_t - W_t$





Theorem: The difference $(M_t - W_t)_{t \geq 0}$ has the same distribution as a reflected Wiener process $(|\tilde{W}_t|)_{t \geq 0}$.

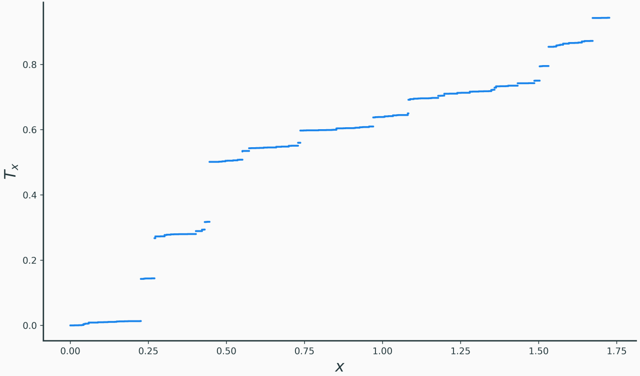
Local time at zero

Consider a clock which runs when $(\tilde{W}_t)_{t\geq 0}$ is zero. This is called the local time at zero.

$$
\begin{aligned}
\text{zeros of } \tilde{W}_t &\leftrightarrow \text{zeros of } |\tilde{W}_t| \\
&\sim \text{zeros of } M_t - W_t \\
&\leftrightarrow \text{times when } W_t = M_t \\
&\leftrightarrow \text{times when } M_t \text{ increases}
\end{aligned}
$$

M_t shows the time $\tilde{W}_t$ has spent at zero!

T_x as a Lévy process



Theorem: $(T_x)_{x\geq 0}$ is a strictly increasing pure-jump Lévy process with transition density

$$p_x(t \mid s) = \frac{x}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{x^2}{2(t-s)}\right), \quad 0 < s < t,\, x > 0.$$

Sketch of the proof

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Sketch: For $x, y > 0$, $(W_t)_{t \geq 0}$ must enter (x, ∞) before $(x + y, \infty)$, meaning $T_x < T_{x+y}$.

Consider the increment $T_{x+y} - T_x$. This equals $\tilde{T}_y$, the hitting time of (y, ∞) for $(\tilde{W}_u := W_{T_x + u} - W_{T_x})_{u \geq 0}$. The strong Markov property implies $\tilde{T}_y$ is independent of T_x.

Pure-jump processes and Poisson random measures

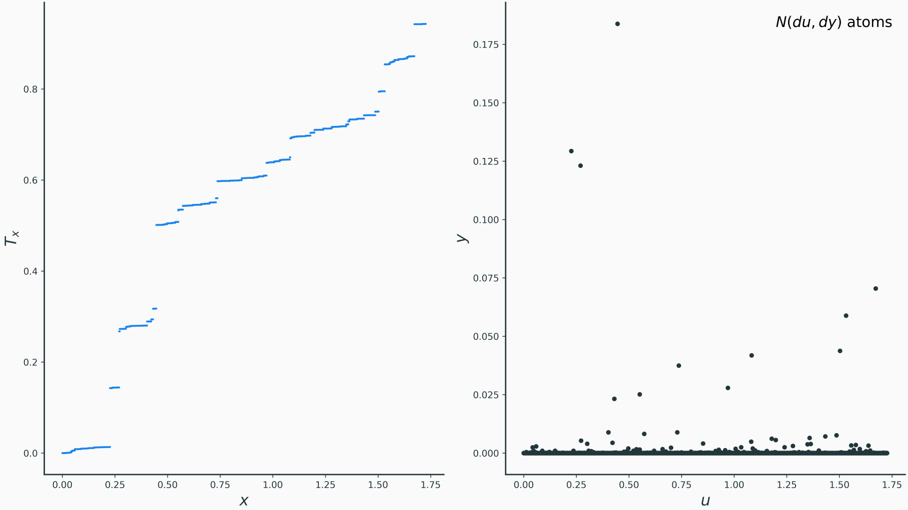
We can write

$$T_x = \int_{[0,x]\times\mathbb{R}_+} N(du, dy)\, y.$$

Here N is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with mean measure $\mathrm{Leb} \otimes \nu$ satisfying

$$\int_{\mathbb{R}_+} \nu(dy)(y \wedge 1) < \infty.$$

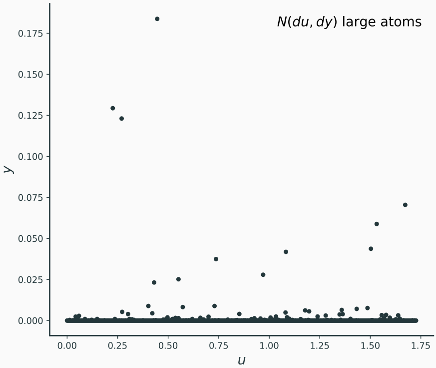
The atoms of N

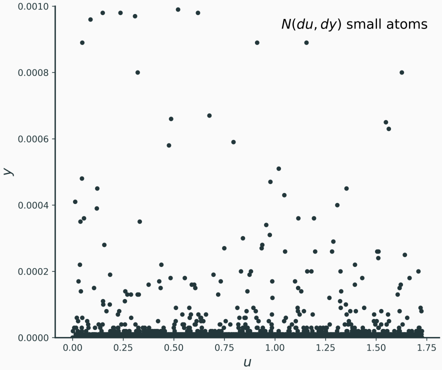


What is the Lévy measure ν?

The Lévy measure

It can be shown $\nu = dy \dfrac{1}{\sqrt{2\pi y^3}}$.





The expected number of atoms in $[0, 1] \times (a, b)$ is

$$(\mathrm{Leb} \otimes \nu)([0, 1] \times (a, b)) = \sqrt{\frac{2}{\pi a}} - \sqrt{\frac{2}{\pi b}}.$$